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Journal of Pure and Applied Algebra 193 (2004) 263–285

JOURNAL OF
PURE AND
APPLIED ALGEBRAwww.elsevier.com/locate/jpaa

Riemann–Roch like theorem for triangulated categories

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Received 30 January 2004; received in revised form 29 February 2004

Communicated by A.V. Geramita

Abstract

We will prove a Riemann–Roch like theorem for triangulated categories satisfying Serre duality. As an application, we will prove Riemann–Roch Theorem and Adjunction Formula for noncommutative Cohen–Macaulay surfaces in terms of sheaf cohomology. We will also show that the formulas hold for stable categories over Koszul connected graded algebras in terms of Tate–Vogel cohomology by extending the BGG correspondence.

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MSC: 14A22; 14C40; 18E30; 14C17; 16S38; 16S37

1. Introduction

Since the classification of curves and surfaces is one of the high points of commutative algebraic geometry, one of the major projects in noncommutative algebraic geometry is to classify noncommutative curves and noncommutative surfaces [22]. Fortunately, one can carry over many tools from commutative algebraic geometry to noncommutative settings. In fact, the classification of noncommutative curves can be regarded as settled by Artin and Stafford [1]. In contrast, the classification of noncommutative surfaces beyond the classification of noncommutative analogues of the projective plane is wide open.

In commutative algebraic geometry, blowing up and intersection theory were essential tools in the classification of surfaces. In [23], Van den Bergh defined *blowing up*

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of a point on a noncommutative surface by introducing notions of quasi-scheme and bimodule over quasi-schemes. On the other hand, *intersection theory* was extended to a quasi-scheme by Jørgensen [9] and Mori–Smith [16] independently. In [9], Jørgensen defined an intersection multiplicity of a module and an effective divisor, which is typically a bimodule so that Tor makes sense, and proved Riemann–Roch Theorem and the genus formula (Adjunction Formula) for classical Cohen–Macaulay surfaces. In [16], we defined an intersection multiplicity of two modules using Ext instead of Tor, and proved Bézout’s Theorem for noncommutative projective spaces (see also [13]). Fortunately, it was proved by Chan [6] that our new intersection multiplicity defined by Ext agrees with the Serre’s intersection multiplicity defined by Tor for commutative locally complete intersection schemes and up to five-dimensional Gorenstein schemes. In this paper, we will prove Riemann–Roch Theorem and Adjunction Formula for classical Cohen–Macaulay surfaces using our intersection multiplicity defined by Ext. Our results are similar to those in [9], but more general since a bimodule can be naturally viewed as a module, but a module may not be naturally viewed as a bimodule. In fact, one of the motivations of this paper is to apply Riemann–Roch Theorem and Adjunction Formula to a fiber of a closed point and a K-theoretic section on a quantum ruled surface defined in [17], which are modules but not bimodules in general.

In the last section, we will extend the BGG correspondence [5] to a Koszul connected graded algebra A such that both A and its Koszul dual $A^!$ are noetherian and having balanced dualizing complexes in the sense of Yekutieli [25], and show that computing Tate–Vogel cohomologies over A is the same as computing sheaf cohomologies over the derived category of the noncommutative projective scheme associated to $A^!$. It follows that Riemann–Roch Theorem and Adjunction Formula hold for stable categories over A in terms of Tate–Vogel cohomology. As a byproduct, we will prove a Serre-like duality for stable categories over AS-Gorenstein Koszul algebras.

To explain the motivation of the paper more in detail, we will now recall some definitions and notations used in noncommutative algebraic geometry. A (commutative) scheme is a pair $X = (X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings \mathcal{O}_X on X , which is locally affine. The category $\text{Mod } X$ of quasi-coherent \mathcal{O}_X -modules is essential to study the scheme X in modern algebraic geometry. It is known that if X is a quasi-compact and quasi-separated scheme, then $\text{Mod } X$ is a Grothendieck category. In [23], Van den Bergh introduced a notion of quasi-scheme in order to develop a theory of noncommutative blowing up.

Definition 1.1. A quasi-scheme is a pair $X = (\text{Mod } X, \mathcal{O}_X)$ where $\text{Mod } X$ is a Grothendieck category and \mathcal{O}_X is an object in $\text{Mod } X$, for which we will write $\mathcal{O}_X \in \text{Mod } X$. A quasi-scheme X is noetherian if $\text{Mod } X$ is locally noetherian, that is, $\text{Mod } X$ has a small set of noetherian generators, and \mathcal{O}_X is a noetherian object. A quasi-scheme over a field k is a quasi-scheme X such that $\text{Mod } X$ is k -linear, that is, Hom_X -set has a k -vector space structure compatible with compositions.

An object in $\text{Mod } X$ is called an X -module. If X is noetherian, then we use a lower letter case $\text{mod } X$ to denote the full subcategory of $\text{Mod } X$ consisting of noetherian objects. For example, if X is a noetherian commutative scheme, then $\text{mod } X$ is the

category of coherent \mathcal{O}_X -modules. An important example of a quasi-scheme for us is a noncommutative projective scheme defined by Artin and Zhang [3].

Definition 1.2. Let A be a finitely generated connected graded algebra over a field k , $\text{Gr Mod } A$ the category of graded right A -modules, and $\text{Fdim } A$ the full subcategory of $\text{Gr Mod } A$ consisting of direct limits of finite dimensional modules. We define the noncommutative projective scheme associated to A to be a quasi-scheme $\text{Proj } A = (\text{Tails } A, \pi_A)$ where $\text{Tails } A = \text{Gr Mod } A / \text{Fdim } A$ is the quotient category and $\pi : \text{Gr Mod } A \rightarrow \text{Tails } A$ is the quotient functor. We often write $\mathcal{M} = \pi M \in \text{Tails } A$ for $M \in \text{Gr Mod } A$.

If A is a finitely generated connected graded domain of GKdimension $d + 1$, then it is reasonable to call $\text{Proj } A$ a noncommutative irreducible projective scheme of dimension d . In fact, noncommutative irreducible projective curves in this sense were classified by Artin and Stafford [1]. Another possible definition of a noncommutative scheme was given in [22], namely, a noncommutative smooth proper scheme of dimension d is a noetherian Ext-finite k -linear abelian category of homological dimension d satisfying Serre duality (see the definitions below). In fact, noncommutative smooth proper curves in this sense were classified by Reiten and Van den Bergh [20]. Our ultimate goal is to classify noncommutative surfaces in either sense. Although the classification of noncommutative surfaces is wide open, in this paper, we will extend the important classical tools in studying commutative surfaces, Riemann–Roch Theorem and Adjunction Formula, to noncommutative surfaces in either sense above.

2. Riemann–Roch theorem for triangulated categories

From now on, we fix a field k , and assume that all categories and functors are k -linear. In order to treat two possible definitions of noncommutative surfaces given in the introduction at the same time, we will first prove Riemann–Roch Theorem and Adjunction Formula for triangulated categories. The following definition is inspired by the definition of a quasi-scheme above.

Definition 2.1. A triangulated quasi-scheme over k is a triple $X = (\text{tri } X, T, \mathcal{O}_X)$ where

- $\text{tri } X$ is a k -linear triangulated category,
- $T : \text{tri } X \rightarrow \text{tri } X$ is the translation functor, and
- $\mathcal{O}_X \in \text{tri } X$.

We refer to [24, Chapter 10] for basic properties of a triangulated category. If $X = (\text{Mod } X, \mathcal{O}_X)$ is a noetherian quasi-scheme over k , then $\text{mod } X$ is an abelian category, so the triple $\mathcal{D}(X) := (\mathcal{D}^b(\text{mod } X), [1], \mathcal{O}_X)$ is a triangulated quasi-scheme over k where $\mathcal{D}^b(\text{mod } X)$ is the bounded derived category of $\text{mod } X$ and $\mathcal{M}[n]^i = \mathcal{M}^{n+i}$ for $\mathcal{M} \in \mathcal{D}^b(\text{mod } X)$ [24, Corollary 10.4.3] (note that, in [24], $\mathcal{M}[n]$ was defined by $\mathcal{M}[n]^i = \mathcal{M}^{i-n}$). If $X = (\text{tri } X, T, \mathcal{O}_X)$ is a triangulated quasi-scheme over k , then so is its opposite $X^o := ((\text{tri } X)^o, T^{-1}, \mathcal{O}_X)$ [24, Exercise 10.2.3].

If X is a triangulated quasi-scheme over k , and $\mathcal{M}, \mathcal{N} \in \text{tri} X$, then we define

$$\text{Ext}_X^i(\mathcal{M}, \mathcal{N}) := \text{Hom}_X(\mathcal{M}, T^i \mathcal{N}).$$

In particular, we define the i th “sheaf cohomology” of \mathcal{M} by

$$H^i(X, \mathcal{M}) := \text{Ext}_X^i(\mathcal{O}_X, \mathcal{M}).$$

We define the Euler form of \mathcal{M} and \mathcal{N} by

$$\xi(\mathcal{M}, \mathcal{N}) := \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \text{Ext}_X^i(\mathcal{M}, \mathcal{N}),$$

and the Euler characteristic of \mathcal{M} by

$$\chi(\mathcal{M}) := \xi(\mathcal{O}_X, \mathcal{M}) = \sum_{i=-\infty}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{M}).$$

Following [16], we define the intersection multiplicity of \mathcal{M} and \mathcal{N} by

$$\mathcal{M} \cdot \mathcal{N} = (-1)^{\text{codim } \mathcal{M}} \xi(\mathcal{M}, \mathcal{N})$$

for some suitably defined integer $\text{codim } \mathcal{M}$. Here, Krull dimension is a natural candidate to be used, however, in practice, Krull dimension is not easy to compute, so we will not specify which dimension function to be used until some concrete examples are considered. Note that $\xi(\mathcal{M}, \mathcal{N})$ is well-defined if and only if

- (1) $\dim_k \text{Ext}_X^i(\mathcal{M}, \mathcal{N}) < \infty$ for all i , and
- (2) $\text{Ext}_X^i(\mathcal{M}, \mathcal{N}) = 0$ for all but finitely many i .

For these conditions, the following definitions are convenient.

Definition 2.2. Let X be a triangulated quasi-scheme over k .

- (1) We say that X is H-finite if $\dim_k H^0(X, \mathcal{M}) = \dim_k \text{Hom}_X(\mathcal{O}_X, \mathcal{M}) < \infty$ for all $\mathcal{M} \in \text{tri} X$.
- (2) We say that X is Hom-finite if $\dim_k \text{Hom}_X(\mathcal{M}, \mathcal{N}) < \infty$ for all pairs $\mathcal{M}, \mathcal{N} \in \text{tri} X$.
- (3) We say that X has finite cohomological dimension if, for each $\mathcal{M} \in \text{tri} X$, $H^i(X, \mathcal{M}) = 0$ for all but finitely many i , that is, the cohomological functor $H^0(X, -)$ is of finite type.
- (4) We say that X has finite homological dimension if, for each pair $\mathcal{M}, \mathcal{N} \in \text{tri} X$, $\text{Ext}_X^i(\mathcal{M}, \mathcal{N}) = 0$ for all but finitely many i .

Let X be a triangulated quasi-scheme. We write $D : \text{tri} X \rightarrow \text{tri} X$ for an exact autoequivalence, $-D : \text{tri} X \rightarrow \text{tri} X$ for the inverse of D , and $\mathcal{M}(D) := D(\mathcal{M}) \in \text{tri} X$ for $\mathcal{M} \in \text{tri} X$. If C and D are exact autoequivalences of $\text{tri} X$, then we define the exact autoequivalence $C + D$ of $\text{tri} X$ by $\mathcal{M}(C + D) := D(C(\mathcal{M}))$. Note that $C + D$ is not isomorphic to $D + C$ in general. The Grothendieck group of X is defined by $K_0(X) := K_0(\text{tri} X)$. (Here we tacitly assume that $\text{tri} X$ is skeletally small.) The image of $\mathcal{M} \in \text{tri} X$ in $K_0(X)$ is denoted by $[\mathcal{M}]$. The following definition is motivated by the definition of a K-theoretic section on a quantum ruled surface [17, Section 5].

Definition 2.3. Let X be a triangulated quasi-scheme. A weak divisor on X is an element $\mathcal{O}_D \in K_0(X)$ of the form $\mathcal{O}_D = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X)$ for some exact auto-equivalence D of $\text{tri} X$. If \mathcal{O}_C and \mathcal{O}_D are weak divisors on X , then \mathcal{O}_{C+D} is a weak divisor defined by $\mathcal{O}_{C+D} = [\mathcal{O}_X] - [\mathcal{O}_X(-C-D)] \in K_0(X)$.

Let X be a triangulated quasi-scheme over k . If X is H-finite and has finite cohomological dimension, then $\chi(\mathcal{M}) = \xi(\mathcal{O}_X, \mathcal{M})$ is well-defined for all $\mathcal{M} \in \text{tri} X$. Since $H^0(X, -) = \text{Hom}_X(\mathcal{O}_X, -) : \text{tri} X \rightarrow \mathbb{Z}$ is a (covariant) cohomological functor [24, Example 10.2.8], if $\mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow T\mathcal{L}$ is an exact triangle in $\text{tri} X$, then there is a long exact sequence

$$\cdots \rightarrow H^i(X, \mathcal{L}) \rightarrow H^i(X, \mathcal{M}) \rightarrow H^i(X, \mathcal{N}) \rightarrow H^{i+1}(X, \mathcal{L}) \rightarrow \cdots.$$

It follows that $\chi(\mathcal{M}) = \chi(\mathcal{L}) + \chi(\mathcal{N})$, so the Euler characteristic induces a linear form

$$\chi(-) : K_0(X) \rightarrow \mathbb{Z}.$$

If \mathcal{O}_D is a weak divisor on X , then

$$\xi(\mathcal{O}_D, \mathcal{M}) = \xi(\mathcal{O}_X, \mathcal{M}) - \xi(\mathcal{O}_X(-D), \mathcal{M}) = \chi(\mathcal{M}) - \chi(\mathcal{M}(D))$$

is well-defined for all $\mathcal{M} \in \text{tri} X$, so it induces a linear form

$$\xi(\mathcal{O}_D, -) : K_0(X) \rightarrow \mathbb{Z}.$$

We define $\text{codim } \mathcal{O}_D = 1$ so that $\mathcal{O}_D \cdot \mathcal{M} = -\xi(\mathcal{O}_D, \mathcal{M})$. (In the language of [9], we assume that all weak divisors we consider satisfy the condition $[(d-1) - \dim]$.) Similarly, if X is Hom-finite and has finite homological dimension, then the Euler form induces a bilinear form

$$\xi(-, -) : K_0(X) \times K_0(X) \rightarrow \mathbb{Z}.$$

The following asymmetric formula is immediate.

Lemma 2.4. Let X be an H-finite triangulated quasi-scheme over k of finite cohomological dimension. If \mathcal{O}_C and \mathcal{O}_D are weak divisors on X , then

$$\xi(\mathcal{O}_C, \mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(C)) - \chi(\mathcal{O}_X(-D)) + \chi(\mathcal{O}_X(-D+C)).$$

In particular,

$$\xi(\mathcal{O}_D, \mathcal{O}_D) = \xi(\mathcal{O}_D, \mathcal{O}_X) + \xi(\mathcal{O}_X, \mathcal{O}_D).$$

Proof. Left to the reader. \square

We will now prove a Riemann–Roch like theorem for Cohen–Macaulay triangulated quasi-schemes defined below.

Definition 2.5. Let X be a triangulated quasi-scheme over k . We say that X is Cohen–Macaulay if

- (1) X is H-finite,
- (2) X has finite cohomological dimension, and

(3) X satisfies Serre duality, that is, there exists $\omega_X \in \text{tri} X$ such that

$$\text{Hom}_X(-, \omega_X) \cong H^0(X, -)^* : \text{tri} X \rightarrow \text{mod } k,$$

where $(-)^*$ denotes the k -vector space dual. We call ω_X the canonical object of X . A Cohen–Macaulay triangulated quasi-scheme X is called Gorenstein if $\omega_X \cong \mathcal{O}_X(K)$ in $\text{tri} X$ for some exact autoequivalence K of $\text{tri} X$. If this is the case, then we define the canonical divisor \mathcal{O}_K on X by $\mathcal{O}_K = [\mathcal{O}_X] - [\mathcal{O}_X(-K)] \in K_0(X)$.

Let \mathcal{C} be a Hom-finite k -linear abelian category. A (right) Serre functor is an autoequivalence (endofunctor) $F : \mathcal{C} \rightarrow \mathcal{C}$ such that

$$\text{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \cong \text{Hom}_{\mathcal{C}}(\mathcal{N}, F(\mathcal{M}))^*$$

for all $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ which are natural in \mathcal{M} and \mathcal{N} [20]. Let X be a Hom-finite triangulated quasi-scheme. If $\text{tri} X$ has a right Serre functor $F : \text{tri} X \rightarrow \text{tri} X$, then X satisfies Serre duality with the canonical object $\omega_X = F(\mathcal{O}_X) \in \text{tri} X$ for any choice of $\mathcal{O}_X \in \text{tri} X$. We will give some examples of a Cohen–Macaulay triangulated quasi-scheme in the subsequent sections.

Note that if a triangulated quasi-scheme X satisfies Serre duality with the canonical object ω_X , then

$$\begin{aligned} \text{Ext}_X^i(-, \omega_X) &= \text{Hom}_X(-, T^i \omega_X) \cong \text{Hom}_X(T^{-i}(-), \omega_X) \\ &\cong H^0(X, T^{-i}(-))^* = H^{-i}(X, -)^* \end{aligned}$$

for all i . So the following lemma is immediate.

Lemma 2.6. *If X is a Cohen–Macaulay triangulated quasi-scheme over k with the canonical object ω_X , then so is $X' := ((\text{tri} X)^o, T^{-1}, \omega_X)$ with the canonical object $\omega_{X'} = \mathcal{O}_X$. Moreover, if X is Gorenstein with the canonical object $\omega_X \cong \mathcal{O}_X(K)$ for some exact autoequivalence K of $\text{tri} X$, then so is X^o with the canonical object $\omega_{X^o} = \mathcal{O}_X(-K)$.*

Proof. Left to the reader. \square

The formulas in the following theorem were called Riemann–Roch and the genus formula in [9, Theorems 5.1, 5.2].

Theorem 2.7. *Suppose that X is a Cohen–Macaulay triangulated quasi-scheme over k , \mathcal{O}_D is a weak divisor on X , and ω_X is the canonical object of X . Then we have the following formulas:*

(1)

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(\mathcal{O}_D \cdot \mathcal{O}_D - \mathcal{O}_D \cdot ([\omega_X] - [\mathcal{O}_X])) + 1 + p_a,$$

where $p_a := \chi(\mathcal{O}_X) - 1$ is the arithmetic genus of X .

(2)

$$2g - 2 = \mathcal{O}_D \cdot \mathcal{O}_D + \mathcal{O}_D \cdot ([\omega_X] - [\mathcal{O}_X]),$$

where $g := 1 - \chi(\mathcal{O}_D)$ is the genus of \mathcal{O}_D .

Proof. Since X is Cohen–Macaulay, for any $\mathcal{M} \in \text{tri } X$, we have

$$\begin{aligned} \zeta(\mathcal{M}, \omega_X) &= \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \text{Ext}_X^i(\mathcal{M}, \omega_X) \\ &= \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \text{Ext}_X^{-i}(\mathcal{O}_X, \mathcal{M})^* \\ &= \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \text{Ext}_X^i(\mathcal{O}_X, \mathcal{M}) \\ &= \zeta(\mathcal{O}_X, \mathcal{M}). \end{aligned}$$

By Lemma 2.4,

$$\begin{aligned} &\mathcal{O}_D \cdot \mathcal{O}_D - \mathcal{O}_D \cdot ([\omega_X] - [\mathcal{O}_X]) \\ &= -\zeta(\mathcal{O}_D, \mathcal{O}_D) + \zeta(\mathcal{O}_D, \omega_X) - \zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= -\zeta(\mathcal{O}_D, \mathcal{O}_X) - \zeta(\mathcal{O}_X, \mathcal{O}_D) + \zeta(\mathcal{O}_X, \mathcal{O}_D) - \zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= -2\zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= -2(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D))) \\ &= -2(1 + p_a) + 2\chi(\mathcal{O}_X(D)), \end{aligned}$$

hence the first formula. Similarly,

$$\begin{aligned} &\mathcal{O}_D \cdot \mathcal{O}_D + \mathcal{O}_D \cdot ([\omega_X] - [\mathcal{O}_X]) \\ &= -\zeta(\mathcal{O}_D, \mathcal{O}_D) - \zeta(\mathcal{O}_D, \omega_X) + \zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= -\zeta(\mathcal{O}_D, \mathcal{O}_X) - \zeta(\mathcal{O}_X, \mathcal{O}_D) - \zeta(\mathcal{O}_X, \mathcal{O}_D) + \zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= -2\zeta(\mathcal{O}_X, \mathcal{O}_D) \\ &= -2\chi(\mathcal{O}_D) \\ &= -2(1 - g) \\ &= 2g - 2, \end{aligned}$$

hence the second formula. \square

If X is a Gorenstein triangulated quasi-scheme over k , then we have formulas which are closer to the classical Riemann–Roch and Adjunction Formula for commutative smooth projective surfaces [7, Chapter V, Theorem 1.6, Proposition 1.5].

Theorem 2.8. *Suppose that X is a Gorenstein triangulated quasi-scheme over k , \mathcal{O}_D is a weak divisor on X , and \mathcal{O}_K is the canonical divisor on X . Then we have the following formulas:*

(1) (Riemann–Roch)

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}(\mathcal{O}_D \cdot \mathcal{O}_D - \mathcal{O}_K \cdot \mathcal{O}_D) + 1 + p_a.$$

(2) (Adjunction formula)

$$2g - 2 = \mathcal{O}_D \cdot \mathcal{O}_D + \mathcal{O}_K \cdot \mathcal{O}_D.$$

Proof. Since

$$\begin{aligned} \mathcal{O}_K \cdot \mathcal{O}_D &= -\zeta(\mathcal{O}_K, \mathcal{O}_D) \\ &= -(\chi(\mathcal{O}_D) - \chi(\mathcal{O}_D(K))) \\ &= -\zeta(\mathcal{O}_X, \mathcal{O}_D) + \zeta(\mathcal{O}_X, \mathcal{O}_D(K)) \\ &= -\zeta(\mathcal{O}_D, \omega_X) + \zeta(\mathcal{O}_D(K), \omega_X) \\ &= -\zeta(\mathcal{O}_D, \omega_X) + \zeta(\mathcal{O}_D(K), \mathcal{O}_X(K)) \\ &= -\zeta(\mathcal{O}_D, \omega_X) + \zeta(\mathcal{O}_D, \mathcal{O}_X) \\ &= \mathcal{O}_D \cdot ([\omega_X] - [\mathcal{O}_X]), \end{aligned}$$

the formulas follow from Theorem 2.7. \square

In our intersection multiplicity defined by Ext, the order may be important. Let X be a Cohen–Macaulay triangulated quasi-scheme over k . We have seen that $\mathcal{O}_D \cdot \omega_X$ is well-defined, but we do not know whether or not $\omega_X \cdot \mathcal{O}_D$ is well-defined unless X is Gorenstein. Even if X is Gorenstein, we do not know whether or not $\mathcal{O}_K \cdot \mathcal{O}_D = \mathcal{O}_D \cdot \mathcal{O}_K$ in general, so we may not be able to simply replace $\mathcal{O}_K \cdot \mathcal{O}_D$ by $\mathcal{O}_D \cdot \mathcal{O}_K$ in the above formulas. In fact, over a noetherian commutative Gorenstein local ring, the commutativity of our intersection multiplicity is equivalent to Serre’s vanishing conjecture (see [13]).

3. Noncommutative Cohen–Macaulay surfaces

In this section, we will prove Riemann–Roch Theorem and Adjunction Formula for noncommutative Cohen–Macaulay surfaces.

Let X be a noetherian quasi-scheme over k , and $\mathcal{M}, \mathcal{N} \in \text{Mod } X$. Since the Grothendieck category $\text{Mod } X$ has enough injectives, \mathcal{N} has an injective resolution

\mathcal{E}^\bullet in $\text{Mod } X$. Since $\text{Hom}_X(\mathcal{M}, -) : \text{Mod } X \rightarrow \text{Mod } k$ is a left exact functor, we can define

$$\text{Ext}_X^i(\mathcal{M}, \mathcal{N}) := h^i(\text{Hom}_X(\mathcal{M}, \mathcal{E}^\bullet)).$$

In particular, we can define the i th “sheaf cohomology” of $\mathcal{M} \in \text{Mod } X$ by

$$H^i(X, \mathcal{M}) := \text{Ext}_X^i(\mathcal{O}_X, \mathcal{M})$$

as before. We say that X is Ext-finite if $\dim_k \text{Ext}_X^i(\mathcal{M}, \mathcal{N}) < \infty$ for all $\mathcal{M}, \mathcal{N} \in \text{mod } X$ and all i . We define the cohomological dimension of X by

$$\text{cd}(X) = \sup\{i \mid H^i(X, \mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{mod } X\},$$

and the homological dimension of X by

$$\text{hd}(X) = \sup\{i \mid \text{Ext}_X^i(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \text{mod } X\}.$$

Note that since $\text{Mod } X$ has enough injectives,

$$\text{Ext}_X^i(\mathcal{M}, \mathcal{N}) \cong \text{Ext}_{\mathcal{D}(X)}^i(\mathcal{M}, \mathcal{N})$$

for all $\mathcal{M}, \mathcal{N} \in \text{mod } X$ and all i [24, Corollary 10.7.5].

Lemma 3.1. *Let X be a noetherian quasi-scheme over k .*

- (1) *If X is Ext-finite, then $\mathcal{D}(X)$ is H-finite.*
- (2) *If $\text{cd}(X) < \infty$, then $\mathcal{D}(X)$ has finite cohomological dimension.*

Proof. Since $\text{Mod } X$ has enough injectives and $\text{Hom}_X(\mathcal{O}_X, -) : \text{Mod } X \rightarrow \text{Mod } k$ is a left exact functor, there is a convergent Grothendieck spectral sequence

$$\begin{aligned} E_2^{pq} &:= \mathbb{R}^p \text{Hom}_X(\mathcal{O}_X, h^q(\mathcal{M})) = \text{Ext}_X^p(\mathcal{O}_X, h^q(\mathcal{M})) \\ &\Rightarrow \mathbb{R}^{p+q} \text{Hom}_X(\mathcal{O}_X, \mathcal{M}) \cong h^{p+q}(\mathbf{R} \text{Hom}_X(\mathcal{O}_X, \mathcal{M})) \\ &\cong \text{Ext}_{\mathcal{D}(X)}^{p+q}(\mathcal{O}_X, \mathcal{M}) = H^{p+q}(\mathcal{D}(X), \mathcal{M}) \end{aligned}$$

for any $\mathcal{M} \in \mathcal{D}^b(\text{mod } X)$ by [24, Corollaries 10.8.3, 10.5.7], hence the result. \square

We will now give some examples of a Cohen–Macaulay triangulated quasi-scheme.

Example 3.2. Let \mathcal{C} be an Ext-finite k -linear abelian category. We say that \mathcal{C} satisfies (right) Serre duality if $\mathcal{D}^b(\mathcal{C})$ has a (right) Serre functor. If \mathcal{C} satisfies right Serre duality, then clearly \mathcal{C} has finite homological dimension. We say that \mathcal{C} is hereditary if homological dimension of \mathcal{C} is at most 1. In [20], Reiten and Van den Bergh classified noetherian Ext-finite hereditary abelian categories over an algebraically closed field k satisfying Serre duality.

If X is a noetherian Ext-finite quasi-scheme over k such that $\text{mod } X$ satisfies (right) Serre duality, then $\text{cd}(X) \leq \text{hd}(X) < \infty$, so $\mathcal{D}(X)$ is H-finite and having finite cohomological dimension by Lemma 3.1. Since $\mathcal{D}(X)$ has a (right) Serre functor, we have seen that $\mathcal{D}(X)$ satisfies Serre duality, hence $\mathcal{D}(X)$ is a Gorenstein (Cohen–Macaulay)

triangulated quasi-scheme. It follows that Riemann–Roch Theorem and Adjunction Formula hold for the derived categories of noncommutative smooth proper schemes defined in the introduction.

Example 3.3. In particular, if X is a noetherian commutative scheme over k , then $\text{cd}(X) \leq \dim X$ by [7, Chapter 3, Theorem 2.7], so $\mathcal{D}(X)$ has finite cohomological dimension. Moreover, if X is projective, then X is Ext-finite and $\text{cd}(X) = \dim X$, so $\mathcal{D}(X)$ is H-finite. In fact, $\mathcal{D}(X)$ is a Cohen–Macaulay triangulated quasi-scheme with $\omega_{\mathcal{D}(X)}$ the dualizing complex of X .

We can similarly construct a Cohen–Macaulay triangulated quasi-scheme from a noncommutative projective scheme.

Definition 3.4. Let A be a noetherian connected graded algebra over k , and $\mathfrak{m} = A_{\geq 1}$ the augmentation ideal of A . For $M, N \in \text{Gr Mod } A$, the set of morphisms in $\text{Gr Mod } A$, that is, right A -module homomorphisms preserving degrees, is denoted by $\text{Hom}_A(M, N)$. We define the functor $\Gamma_{\mathfrak{m}} : \mathcal{D}(\text{Gr Mod } A) \rightarrow \mathcal{D}(\text{Gr Mod } A)$ by

$$\Gamma_{\mathfrak{m}}(-) := \lim_{n \rightarrow \infty} \left(\bigoplus_{m=-\infty}^{\infty} \text{Hom}_A(A/A_{\geq n}, (-)(m)) \right).$$

The right derived functor of $\Gamma_{\mathfrak{m}}$ is denoted by $\mathbf{R}\Gamma_{\mathfrak{m}}$, and the i th local cohomology functor is defined by

$$H_{\mathfrak{m}}^i(-) := h^i(\mathbf{R}\Gamma_{\mathfrak{m}}(-)) = \lim_{n \rightarrow \infty} \left(\bigoplus_{m=-\infty}^{\infty} \text{Ext}_A^i(A/A_{\geq n}, (-)(m)) \right).$$

Similarly, we define the functor $\Gamma_{\mathfrak{m}^o} : \mathcal{D}(\text{Gr Mod } A^o) \rightarrow \mathcal{D}(\text{Gr Mod } A^o)$ where A^o is the opposite graded algebra of A . Here, we identify a graded left A -module with a graded right A^o -module. We define the depth of $M \in \text{Gr Mod } A$ by

$$\text{depth } M = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\},$$

and the local dimension of $M \in \text{Gr Mod } A$ by

$$\text{ldim } M = \sup\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

A balanced dualizing complex was first introduced by Yekutieli [25].

Definition 3.5. Let A be a noetherian connected graded algebra over k , and $A^e := A \otimes_k A^o$. We identify a graded A – A bimodule with a graded A^e -module. An object $R \in \mathcal{D}^b(\text{Gr Mod } A^e)$ is called a balanced dualizing complex if

- R has finite injective dimension over A and A^o ,
- $h^i(R)$ are finitely generated over A and A^o for all i ,
- the natural morphisms

$$A \rightarrow \mathbf{R} \left(\bigoplus_{m=-\infty}^{\infty} \text{Hom}_A(R, R(m)) \right)$$

and

$$A \rightarrow \mathbf{R} \left(\bigoplus_{m=-\infty}^{\infty} \mathrm{Hom}_{A^e}(R, R(m)) \right)$$

are isomorphisms in $\mathcal{D}(\mathrm{Gr} \mathrm{Mod} A^e)$, and

- $\mathbf{R}\Gamma_{\mathfrak{m}}(R) \cong \mathbf{R}\Gamma_{\mathfrak{m}^e}(R) \cong A^*$ in $\mathcal{D}(\mathrm{Gr} \mathrm{Mod} A^e)$ where A^* is a graded k -vector space dual of A endowed with the natural graded A – A bimodule structure.

A balanced dualizing complex is unique up to isomorphisms in $\mathcal{D}(\mathrm{Gr} \mathrm{Mod} A^e)$ if it exists.

Lemma 3.6. *If A is a noetherian connected graded algebra over k having the balanced dualizing complex R , then $X = \mathcal{D}(\mathrm{Proj} A)$ is a Cohen–Macaulay triangulated quasi-scheme with the canonical object $\omega_X = \pi R[-1] \in \mathcal{D}^b(\mathrm{tails} A)$.*

Proof. This follows from Lemma 3.1 and [26, Theorem 4.2.2]. \square

Let X be a noetherian quasi-scheme. We can define a weak divisor on X by $\mathcal{O}_D = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X) := K_0(\mathrm{mod} X)$ for some autoequivalence D of $\mathrm{mod} X$ as before. Since any autoequivalence $D : \mathrm{mod} X \rightarrow \mathrm{mod} X$ induces an exact autoequivalence $D : \mathcal{D}^b(\mathrm{mod} X) \rightarrow \mathcal{D}^b(\mathrm{mod} X)$, it follows that $\mathcal{O}_D = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(\mathcal{D}(X))$ is a weak divisor on $\mathcal{D}(X)$ in the sense of Definition 2.3. Note that if X is a noetherian commutative scheme and D is an effective divisor on X in the usual sense, then there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

in $\mathrm{mod} X$ where $\mathcal{O}_X(-D)$ is an invertible \mathcal{O}_X -module and \mathcal{O}_D is the structure sheaf of D , so $[\mathcal{O}_D] = [\mathcal{O}_X] - [\mathcal{O}_X(-D)] \in K_0(X)$. Since $-\otimes_X \mathcal{O}_X(D) : \mathrm{mod} X \rightarrow \mathrm{mod} X$ is an autoequivalence of $\mathrm{mod} X$, \mathcal{O}_D is a weak divisor on X in the above sense.

There was a notion of classical Cohen–Macaulay for a quasi-scheme defined by Yekutieli and Zhang [26].

Definition 3.7. Let X be a noetherian quasi-scheme over k . We say that X is classical Cohen–Macaulay if

- X is Ext-finite,
- $\mathrm{cd}(X) = d < \infty$, and
- X satisfies classical Serre duality, that is, there exists an object $\omega_X \in \mathrm{mod} X$ such that

$$\mathrm{Ext}_X^i(-, \omega_X) \cong \mathrm{H}^{d-i}(X, -)^* : \mathrm{mod} X \rightarrow \mathrm{mod} k$$

for all i .

We call ω_X the canonical sheaf on X . A classical Cohen–Macaulay quasi-scheme X is called classical Gorenstein if $\omega_X \cong \mathcal{O}_X(K)$ in $\mathrm{mod} X$ for some autoequivalence

K of $\text{mod } X$. If this is the case, then we define the canonical divisor \mathcal{O}_K on X by $\mathcal{O}_K = [\mathcal{O}_X] - [\mathcal{O}_X(-K)] \in K_0(X)$.

We will now give some examples of a classical Cohen–Macaulay quasi-scheme. The following algebras have been intensively studied in noncommutative algebraic geometry. Here AS stands for Artin–Schelter.

Definition 3.8. Let A be a noetherian connected graded algebra over k having the balanced dualizing complex.

- We say that A is AS–Cohen–Macaulay if $\text{depth } A = \text{ldim } A$.
- We say that A is AS–Gorenstein if $\text{id}(A) < \infty$.
- We say that A is AS–regular if $\text{gldim } A < \infty$.

Example 3.9. Let A be a noetherian connected graded algebra over k having the balanced dualizing complex R . Then A is AS–Cohen–Macaulay if and only if $R \cong \omega_A[d]$ in $\mathcal{D}(\text{Gr Mod } A^e)$ for some graded A – A bimodule ω_A , which we call the canonical module of A , and $d = \text{depth } A$. If this is the case, then $X = \text{Proj } A$ is a classical Cohen–Macaulay quasi-scheme with the canonical sheaf $\omega_X = \pi\omega_A \in \text{tails } A$, and $\mathcal{D}(X)$ is a Cohen–Macaulay triangulated quasi-scheme with the canonical object $\omega_{\mathcal{D}(X)} = \pi\omega_A[d-1] \in \mathcal{D}^b(\text{tails } A)$ by Lemma 3.6. Moreover, A is AS–Gorenstein if and only if $\omega_A \cong A(-\ell)$ in $\text{grmod } A$ for some integer ℓ . If this is the case, then $X = \text{Proj } A$ is classical Gorenstein with the canonical sheaf $\omega_X = \mathcal{A}(-\ell) \in \text{tails } A$, and $\mathcal{D}(X)$ is Gorenstein with the canonical object $\omega_{\mathcal{D}(X)} = \mathcal{A}[d-1](-\ell)$.

Example 3.10. In particular, a quantum projective space over k , which is typically of the form $X = \text{Proj } A$ for some AS–regular algebra A , is classical Gorenstein having finite homological dimension. Moreover, a quantum ruled surface over a commutative smooth projective scheme of finite type over k is classical Cohen–Macaulay by [18, Theorems 4.16, 5.20]; [19, Corollary 3.6], and having finite homological dimension by [17, Proposition 5.4]. We refer to [16,17] for intersection theory over a quantum projective space and a quantum ruled surface, respectively.

As stated in the introduction, our aim is to use intersection theory to classify noncommutative surfaces. We will now show that Riemann–Roch Theorem and Adjunction Formula hold for noncommutative Cohen–Macaulay surfaces.

Theorem 3.11. Let X be a noetherian classical Cohen–Macaulay quasi-scheme over k . If $\text{cd}(X) = d$ is an even integer (e.g. $\text{cd}(X) = 2$ so that X is a “noncommutative Cohen–Macaulay surface”), then exactly the same formulas in Theorem 2.7 hold for X . Moreover, if X is classical Gorenstein, then exactly the same formulas in Theorem 2.8 hold for X .

Proof. If ω_X is the canonical sheaf on X , then

$$\text{Hom}_{\mathcal{D}(X)}(\mathcal{M}, \omega_X[d]) \cong \text{Ext}_X^d(\mathcal{M}, \omega_X) \cong H^0(X, \mathcal{M})^* \cong H^0(\mathcal{D}(X), \mathcal{M})^*$$

for all $\mathcal{M} \in \text{mod } X$, so $\omega_X[d] \in \mathcal{D}^b(\text{mod } X)$ behaves like a canonical object of $\mathcal{D}(X)$ with respect to objects in $\text{mod } X$. Since d is an even integer,

$$\begin{aligned}\mathcal{O}_D \cdot \omega_X[d] &= -\xi(\mathcal{O}_D, \omega_X[d]) \\ &= -(-1)^d \xi(\mathcal{O}_D, \omega_X) \\ &= -\xi(\mathcal{O}_D, \omega_X) \\ &= \mathcal{O}_D \cdot \omega_X.\end{aligned}$$

It follows that exactly the same proofs of Theorem 2.7 and Theorem 2.8 will prove the results. \square

If $\text{cd}(X) = d$ is an odd integer in the above theorem, then the similar formulas in Theorem 2.7, replacing $[\omega_X]$ by $-[\omega_X]$, hold for X . However, we do not know any nice formulas for X corresponding to those in Theorem 2.8 in this case. If A is a reasonably nice AS–Cohen–Macaulay algebra such as PI, then $\text{cd}(\text{Proj } A) = \text{GKdim } A - 1$, so the above Theorem typically applies to noncommutative Cohen–Macaulay projective surfaces defined in the introduction. Note that since our intersection multiplicity agrees with the Serre’s intersection multiplicity for noetherian commutative Gorenstein surfaces [6], even applying to commutative surfaces, the above theorem is more general than the usual Riemann–Roch Theorem and Adjunction Formula given in [7, Chapter V, Theorem 1.6, Proposition 1.5].

4. Comparison with intersection theory by Jørgensen

There is another intersection theory for noncommutative surfaces developed by Jørgensen [9]. In this section, we will compare these two intersection theories.

In [23], Van den Bergh introduced a notion of bimodule over quasi-schemes. Let X, Y, Z be quasi-schemes. We define the following categories:

- $\text{Lex}(Y, X)$ = the category of left exact functors $\text{Mod } Y \rightarrow \text{Mod } X$.
- $\text{BIMOD}(X, Y)$ = the opposite category of $\text{Lex}(Y, X)$.
- $\text{BiMod}(X, Y)$ = the full subcategory of $\text{BIMOD}(X, Y)$ consisting of objects having left adjoints, that is, an object in $\text{BiMod}(X, Y)$ is an adjoint pair of functors.

An object $\mathcal{M} \in \text{BIMOD}(X, Y)$ is called a weak X – Y bimodule, and the corresponding left exact functor is denoted by $\mathcal{H}om_Y(\mathcal{M}, -) : \text{Mod } Y \rightarrow \text{Mod } X$. An object $\mathcal{M} \in \text{BiMod}(X - Y)$ is called an X – Y bimodule, and the corresponding right exact functor is denoted by $- \otimes_X \mathcal{M} : \text{Mod } X \rightarrow \text{Mod } Y$, so that

$$\text{Hom}_Y(- \otimes_X \mathcal{M}, -) \cong \text{Hom}_X(-, \mathcal{H}om_Y(\mathcal{M}, -)).$$

If $\mathcal{M} \in \text{BIMOD}(X, Y)$ and $\mathcal{N} \in \text{BIMOD}(Y, Z)$, then the composition of \mathcal{M} and \mathcal{N} is denoted by $\mathcal{M} \otimes_Y \mathcal{N} \in \text{BIMOD}(X, Z)$, so that

$$\mathcal{H}om_Z(\mathcal{M} \otimes_Y \mathcal{N}, -) \cong \mathcal{H}om_Y(\mathcal{M}, \mathcal{H}om_Z(\mathcal{N}, -)).$$

An X – Y bimodule \mathcal{M} is called noetherian if the functors $-\otimes_X \mathcal{M} : \text{Mod } X \rightarrow \text{Mod } X$ and $\mathcal{H}om_Y(\mathcal{M}, -) : \text{Mod } Y \rightarrow \text{Mod } X$ send noetherian objects to noetherian objects. We write o_X for the identity functor $o_X : \text{Mod } X \rightarrow \text{Mod } X$, viewed as an X – X bimodule. An X – Y bimodule \mathcal{L} is invertible if there is an Y – X bimodule \mathcal{M} such that $\mathcal{L} \otimes_Y \mathcal{M} \cong o_X$ and $\mathcal{M} \otimes_X \mathcal{L} \cong o_Y$. Invertible bimodules are clearly noetherian. We often write $o_X(D)$ for an invertible X – X bimodule, $o_X(-D)$ for the inverse of $o_X(D)$, and $\mathcal{N}(D) := \mathcal{N} \otimes_X o_X(D) \in \text{Mod } X$ for $\mathcal{N} \in \text{Mod } X$, so that these notations agree with the notations defined in the previous sections.

Let X, Y be quasi-schemes. If $\mathcal{M} \in \text{BIMOD}(X, Y)$, then $\mathcal{H}om_Y(\mathcal{M}, -) : \text{Mod } Y \rightarrow \text{Mod } X$ is a left exact functor, so, for $\mathcal{N} \in \text{Mod } Y$, we can define

$$\mathcal{E}xt_Y^i(\mathcal{M}, \mathcal{N}) := h^i(\mathcal{H}om_Y(\mathcal{M}, \mathcal{E}^\bullet)),$$

where \mathcal{E}^\bullet is an injective resolution of \mathcal{N} in $\text{Mod } Y$. Moreover, for $\mathcal{L} \in \text{Mod } X$, we can also define $\mathcal{T}or_i^X(\mathcal{L}, \mathcal{M})$ to be a unique object up to isomorphisms in $\text{Mod } Y$ satisfying

$$\text{Hom}_Y(\mathcal{T}or_i^X(\mathcal{L}, \mathcal{M}), \mathcal{E}) \cong \text{Ext}_X^i(\mathcal{L}, \mathcal{H}om_Y(\mathcal{M}, \mathcal{E}))$$

for every injective object $\mathcal{E} \in \text{Mod } Y$ [17, Lemma 1.4].

An effective divisor on a quasi-scheme was defined in [23].

Definition 4.1. Let X be a noetherian quasi-scheme over k . An effective divisor on X is an X – X bimodule o_D which admits an exact sequence

$$0 \rightarrow o_X(-D) \rightarrow o_X \rightarrow o_D \rightarrow 0$$

in $\text{BIMOD}(X, X)$ for some invertible X – X bimodule $o_X(D)$. An effective divisor o_D on X is called admissible if $\mathcal{H}om_X(o_D, \mathcal{O}_X) = 0$.

Note that if o_D is an effective divisor on X , then $o_X(-D), o_X, o_D \in \text{BiMod}(X, X)$ by [9, Lemma 1.5].

Let X be a noetherian Ext-finite quasi-scheme over k of finite cohomological dimension and o_D an effective divisor on X . In [9], Jørgensen defined the first Chern class of o_D by an endomorphism

$$c(D) : K_0(X) \rightarrow K_0(X); [\mathcal{M}] \mapsto [\mathcal{M}] - [\mathcal{M}(-D)],$$

and the intersection multiplicity of $[\mathcal{M}] \in K_0(X)$ and o_D by

$$\langle D, [\mathcal{M}] \rangle := \chi(c(D)[\mathcal{M}]).$$

His definition of intersection multiplicity can be extended to a general X -module \mathcal{M} and a general X – X bimodule \mathcal{N} as

$$\langle \mathcal{N}, \mathcal{M} \rangle := \sum_{i=0}^{\infty} (-1)^i \chi(\mathcal{T}or_i^X(\mathcal{M}, \mathcal{N})).$$

In fact, applying the functor $\mathcal{M} \otimes_X -$ to the exact sequence

$$0 \rightarrow o_X(-D) \rightarrow o_X \rightarrow o_D \rightarrow 0$$

in $\text{BIMOD}(X, X)$, we have an exact sequence

$$0 \rightarrow \mathcal{T}or_1^X(\mathcal{M}, o_D) \rightarrow \mathcal{M}(-D) \rightarrow \mathcal{M} \rightarrow \mathcal{M} \otimes_X o_D \rightarrow 0$$

in $\text{mod } X$ and $\mathcal{T}or_i^X(\mathcal{M}, o_D) = 0$ for $i \geq 2$, so

$$\begin{aligned} \langle D, \mathcal{M} \rangle &:= \chi(c(D)[\mathcal{M}]) \\ &= \chi([\mathcal{M}] - [\mathcal{M}(-D)]) \\ &= \chi(\mathcal{M}) - \chi(\mathcal{M}(-D)) \\ &= \chi(\mathcal{M} \otimes_X o_D) - \chi(\mathcal{T}or_1^X(\mathcal{M}, o_D)) \\ &= \sum_{i=0}^{\infty} (-1)^i \chi(\mathcal{T}or_i^X(\mathcal{M}, o_D)) \\ &=: \langle o_D, \mathcal{M} \rangle. \end{aligned}$$

This shows that his definition of intersection multiplicity looks more like the Serre's intersection multiplicity for commutative schemes. He proved Riemann–Roch Theorem and the genus formula (Adjunction Formula) for admissible effective divisors on a non-commutative Cohen–Macaulay surface using his definition of intersection multiplicity [9, Theorems 5.1, 5.2]. Note that he defined and assumed some technical conditions, such as [Fixd comp] and [Invariant], to prove his results. This already shows an advantage of Theorem 3.11 because those technical conditions are presumably difficult to check (see [9, Example 5.5] for a simple case). We will also see below that Theorem 3.11 is in fact more general.

Lemma 4.2. *Let X be a noetherian quasi-scheme. If o_D is an admissible effective divisor on X , then $\mathcal{O}_D := \mathcal{O}_X \otimes_X o_D$ is a weak divisor.*

Proof. Let o_D be an admissible effective divisor on X . Applying the functor $\mathcal{H}om_X(-, \mathcal{O}_X)$ to the exact sequence

$$0 \rightarrow o_D \rightarrow o_X \rightarrow o_X(-D) \rightarrow 0$$

in $\text{BIMOD}(X, X)$, we have an exact sequence

$$0 \rightarrow \mathcal{H}om_X(o_D, \mathcal{O}_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{E}xt_X^1(o_D, \mathcal{O}_X) \rightarrow 0$$

in $\text{mod } X$. Further, applying the functor $- \otimes_X o_X(-D)$, we have an exact sequence

$$0 \rightarrow \mathcal{H}om_X(o_D, \mathcal{O}_X)(-D) \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}xt_X^1(o_D, \mathcal{O}_X)(-D) \rightarrow 0$$

in $\text{mod } X$. Since o_D is admissible,

$$\mathcal{T}or_1^X(\mathcal{O}_X, o_D) \cong \mathcal{H}om_X(o_D, \mathcal{O}_X)(-D) = 0,$$

so there is an exact sequence

$$0 = \mathcal{T}or_1^X(\mathcal{O}_X, o_D) \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \otimes_X o_D \rightarrow 0$$

in $\text{mod } X$, hence $\mathcal{O}_D := \mathcal{O}_X \otimes_X o_D \cong \mathcal{E}xt_X^1(o_D, \mathcal{O}_X)(-D)$ is a weak divisor. \square

A weak divisor is presumably much more general than an effective divisor, as the following examples show.

Example 4.3. Let A be a noetherian connected graded domain, so that $X = \text{Proj } A$ is a noetherian noncommutative irreducible projective scheme in the sense of the introduction. If $M = A/fA$ for some homogeneous element $0 \neq f \in A_m$, then $\mathcal{M} = \pi M \in \text{tails } A$ should be considered as (the structure sheaf of) a “hypersurface” on X defined by f (see [16]). Since the exact sequence

$$0 \rightarrow A(-m) \xrightarrow{f} A \rightarrow M \rightarrow 0$$

in $\text{grmod } A$ induces an exact sequence

$$0 \rightarrow \mathcal{O}_X(-m) \rightarrow \mathcal{O}_X \rightarrow \mathcal{M} \rightarrow 0$$

in $\text{mod } X$, \mathcal{M} is a weak divisor on X . However, it is not clear whether or not there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

in $\text{BIMOD}(X, X)$ which induces the exact sequence

$$0 \rightarrow \mathcal{O}_X(-D) \cong \mathcal{O}_X(-m) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X \otimes_X \mathcal{O}_D \cong \mathcal{M} \rightarrow 0$$

in $\text{mod } X$ unless f is a normalizing element of A .

Example 4.4. Let A be an AS-Gorenstein algebra, so that $\text{Proj } A$ is classical Gorenstein. It is not clear whether or not the canonical divisor \mathcal{O}_K on $\text{Proj } A$ is an effective divisor in general. If A is an AS-regular algebra of $\text{gldim } A \leq 3$, then the canonical divisor \mathcal{O}_K is in fact an effective divisor on $\text{Proj } A$ by [2, Theorem 6.8.(i)].

Example 4.5. In [17, Section 5], we defined a K-theoretic section H on a quantum ruled surface. By definition, H is a weak divisor, but it is not clear whether or not H is an effective divisor in general.

5. Stable categories

In this last section, we will prove Riemann–Roch Theorem and Adjunction Formula for stable categories over Koszul connected graded algebras in terms of Tate–Vogel cohomology by extending the BGG correspondence.

First, we will recall the definition of the stabilization of a left triangulated category. Let \mathcal{C} be a left triangulated category. The stabilization of \mathcal{C} is a pair $(S, \mathcal{S}(\mathcal{C}))$, where $\mathcal{S}(\mathcal{C})$ is a triangulated category and $S : \mathcal{C} \rightarrow \mathcal{S}(\mathcal{C})$ is an exact functor such that for any exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to a triangulated category \mathcal{D} , there exists a unique exact functor $\bar{F} : \mathcal{S}(\mathcal{C}) \rightarrow \mathcal{D}$ such that $\bar{F} \circ S = F$. We refer to [4] for basic properties of the stabilization.

Let A be a noetherian connected graded algebra over k . We denote $\underline{\text{grmod}} A$ for the stable category of $\underline{\text{grmod}} A$ by projectives. For $M, N \in \underline{\text{grmod}} A$, the set of morphisms

in $\text{grmod } A$ is denoted by $\underline{\text{Hom}}_A(M, N)$. We denote $\Omega^i M \in \text{grmod } A$ for the i th syzygy of M . The i th Tate–Vogel cohomology is defined by

$$\underline{\text{Ext}}_A^i(M, N) := \lim_{n \rightarrow \infty} \underline{\text{Hom}}_A(\Omega^{n+i} M, \Omega^n N).$$

It is known that the stable category $\text{grmod } A$ is a left triangulated category, so we can define its stabilization $\mathcal{S}(\text{grmod } A)$. We will now quote two results from [4].

Lemma 5.1. *If A is a noetherian connected graded algebra and \mathcal{P} is the full subcategory of $\text{grmod } A$ consisting of projective modules, then there is an equivalence*

$$G : \mathcal{D}^b(\text{grmod } A) / \mathcal{D}^b(\mathcal{P}) \rightarrow \mathcal{S}(\text{grmod } A)$$

as triangulated categories.

Proof. This is a special case of [4, Corollary 3.9]. \square

Lemma 5.2. *If A is a noetherian connected graded algebra and $M, N \in \text{grmod } A$, then*

$$\underline{\text{Ext}}_A^i(M, N) \cong \underline{\text{Ext}}_{\mathcal{S}(\text{grmod } A)}^i(S(M), S(N))$$

for all i .

Proof. See [4, Section 5]. \square

Since the degree shift functor $(n) : \text{grmod } A \rightarrow \text{grmod } A$ naturally induces an autoequivalence $(n) : \mathcal{D}^b(\text{grmod } A) / \mathcal{D}^b(\mathcal{P}) \rightarrow \mathcal{D}^b(\text{grmod } A) / \mathcal{D}^b(\mathcal{P})$, we can define an autoequivalence $(n) : \mathcal{S}(\text{grmod } A) \rightarrow \mathcal{S}(\text{grmod } A)$ via the equivalence functor G . By construction of the functor G in [4, Corollary 3.9], $S(M(n)) \cong S(M)(n)$ for all $M \in \text{grmod } A$ and all n .

Recall that a finitely generated connected graded algebra A over k is called Koszul if $k = A/\mathfrak{m}$ has a linear resolution, that is, a free resolution of the form

$$\cdots \rightarrow \oplus A(-2) \rightarrow \oplus A(-1) \rightarrow A \rightarrow k \rightarrow 0$$

in $\text{GrMod } A$. If A is Koszul, then $A \cong T(V)/(R)$ where V is a finite dimensional k -vector space, $T(V)$ is the tensor algebra over V and $R \subset V \otimes_k V$ is a subvector space. In this case, we define the Koszul dual of A by $A^! := T(V^*)/(R^\perp)$ where

$$R^\perp = \{\lambda \in V^* \otimes_k V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

Note that if A is Koszul, then so is $A^!$, and $(A^!)^! \cong A$ as graded algebras.

In order to prove the next theorem on Koszul algebras, we will define some more notations. Let A be a noetherian connected graded algebra over k . For $M, N \in \mathcal{D}^b(\text{grmod } A)$, we define

$$P_A^{M,N}(s, t) := \sum_{p,q} \dim_k \text{Ext}_A^p(M, N(q)) s^p t^q \in \mathbb{Z}[[s, s^{-1}, t, t^{-1}]].$$

Then the Hilbert series of M is defined by

$$H_M(t) := P_A^{A,M}(-1, t) = \sum_{p,q} (-1)^p \dim_k h^p(M)_q t^q \in \mathbb{Z}[[t, t^{-1}]],$$

and the Poincaré series of M is defined by

$$P_A^M(t) := P_A^{M,k}(t, 1) = \sum_{p,q} \dim_k \operatorname{Ext}_A^p(M, k(q)) t^p \in \mathbb{Z}[[t, t^{-1}]].$$

The following theorem, partially announced in [15], extends the formula in [10, Theorem 6] to non-Koszul modules.

Theorem 5.3. *Let A be a Koszul connected graded algebra such that both A and $A^!$ are noetherian and having balanced dualizing complexes. Then there is a duality*

$$F : \mathcal{D}^b(\operatorname{tails} A) \rightarrow \mathcal{S}(\operatorname{grmod} A^!)$$

as triangulated categories, having the property

$$F(\mathcal{M}[p](q)) = \Omega^{p+q} F(\mathcal{M})(q)$$

for all $\mathcal{M} \in \mathcal{D}^b(\operatorname{tails} A)$ and all p, q . In particular, if $\mathcal{M}, \mathcal{N} \in \operatorname{tails} A$, then

$$\operatorname{Ext}_{\operatorname{tails} A}^p(\mathcal{M}, \mathcal{N}(q)) \cong \operatorname{Ext}_{\mathcal{S}(\operatorname{grmod} A^!)}^{p+q}(F(\mathcal{N})(q), F(\mathcal{M}))$$

for all p, q , and if $M', N' \in \operatorname{grmod} A^!$, then

$$\underline{\operatorname{Ext}}_{A^!}^p(M', N'(q)) \cong \operatorname{Ext}_{\mathcal{D}^b(\operatorname{tails} A)}^{p+q}(F^{-1}(N')(q), F^{-1}(M'))$$

for all p, q .

Proof. By [14, Proposition 4.5], there is a duality

$$\tilde{E} : \mathcal{D}^b(\operatorname{grmod} A) \rightarrow \mathcal{D}^b(\operatorname{grmod} A^!)$$

as triangulated categories. Let $M \in \mathcal{D}^b(\operatorname{grmod} A)$. Since $A^!$ is noetherian, $\tilde{E}(M)$ has a finitely generated minimal free resolution, so $P_{A^!}^{\tilde{E}(M)}(t)$ is well-defined. By [14, Lemma 6.2],

$$\begin{aligned} P_{A^!}^{\tilde{E}(M)}(t) &= P_{A^!}^{\tilde{E}(M),k}(t, 1) \\ &= P_A^{A,M}(t, t) \\ &= \sum_{p,q} \dim_k \operatorname{Ext}_A^p(A, M(q)) t^p t^q \\ &= \sum_p \left(\sum_q \dim_k h^p(M)_q t^q \right) t^p \\ &= \sum_{p:\text{finite}} H_{h^p(M)}(t) t^p. \end{aligned}$$

Let $\mathcal{D}_{\operatorname{fdim} A}^b(\operatorname{grmod} A)$ be the full subcategory of $\mathcal{D}^b(\operatorname{grmod} A)$ generated by complexes whose cohomologies are all in $\operatorname{fdim} A$, and $\mathcal{P}^!$ the full subcategory of $\operatorname{grmod} A^!$ consisting of projective modules. Since coefficients of $H_{h^p(M)}(t)$ are all nonnegative integers

for all p ,

$$\begin{aligned}
 \bar{E}(M) \in \mathcal{D}^b(\mathcal{P}^1) &\Leftrightarrow \text{pd } \bar{E}(M) < \infty \\
 &\Leftrightarrow P_{A^1}^{\bar{E}(M)}(t) \in \mathbb{Z}[t, t^{-1}] \\
 &\Leftrightarrow H_{h^p(M)}(t) \in \mathbb{Z}[t, t^{-1}] \quad \text{for all } p \\
 &\Leftrightarrow h^p(M) \in \text{fdim } A \quad \text{for all } p \\
 &\Leftrightarrow M \in \mathcal{D}_{\text{fdim } A}^b(\text{grmod } A).
 \end{aligned}$$

By [12, Theorem 3.2], there is an equivalence

$$\mathcal{D}^b(\text{tails } A) \cong \mathcal{D}^b(\text{grmod } A) / \mathcal{D}_{\text{fdim } A}^b(\text{grmod } A)$$

as triangulated categories, so \bar{E} induces a duality

$$\bar{E} : \mathcal{D}^b(\text{tails } A) \cong \mathcal{D}^b(\text{grmod } A) / \mathcal{D}_{\text{fdim } A}^b(\text{grmod } A) \rightarrow \mathcal{D}^b(\text{grmod } A^1) / \mathcal{D}^b(\mathcal{P}^1)$$

as triangulated categories. By Lemma 5.1, there is an equivalence

$$G : \mathcal{D}^b(\text{grmod } A^1) / \mathcal{D}^b(\mathcal{P}^1) \rightarrow \mathcal{S}(\text{grmod } A^1)$$

as triangulated categories, so the composition $F = G \circ \bar{E} : \mathcal{D}^b(\text{tails } A) \rightarrow \mathcal{S}(\text{grmod } A^1)$ gives a desired duality as triangulated categories.

By [14, Lemma 2.7],

$$\begin{aligned}
 F(\mathcal{M}[p](q)) &= G(\bar{E}(\mathcal{M}[p](q))) \\
 &= G(\bar{E}(\mathcal{M})[-p-q](q)) \\
 &= \Omega^{p+q} G(\bar{E}(\mathcal{M}))(q) \\
 &= \Omega^{p+q} F(\mathcal{M})(q)
 \end{aligned}$$

for all $\mathcal{M} \in \mathcal{D}^b(\text{tails } A)$ and all p, q . It follows that

$$\begin{aligned}
 \text{Ext}_{\text{tails } A}^p(\mathcal{M}, \mathcal{N}(q)) &\cong \text{Ext}_{\mathcal{D}^b(\text{tails } A)}^p(\mathcal{M}, \mathcal{N}(q)) \\
 &= \text{Hom}_{\mathcal{D}^b(\text{tails } A)}(\mathcal{M}, \mathcal{N}(q)[p]) \\
 &\cong \text{Hom}_{\mathcal{S}(\text{grmod } A^1)}(F(\mathcal{N}(q)[p]), F(\mathcal{M})) \\
 &\cong \text{Hom}_{\mathcal{S}(\text{grmod } A^1)}(\Omega^{p+q} F(\mathcal{N})(q), F(\mathcal{M})) \\
 &= \text{Ext}_{\mathcal{S}(\text{grmod } A^1)}^{p+q}(F(\mathcal{N})(q), F(\mathcal{M}))
 \end{aligned}$$

for all $\mathcal{M}, \mathcal{N} \in \text{tails } A$ and all p, q . By Lemma 5.2, the last formula follows similarly. \square

The above theorem can be thought of as a vast generalization of the classical result known as the BGG correspondence [5], which asserts the equivalence below for a polynomial algebra A (see [11] for another direction of generalization).

Corollary 5.4. *Let A be an AS-regular Koszul algebra, then there is an equivalence*

$$\mathcal{D}^b(\text{tails } A) \cong \underline{\text{grmod}}(A^\perp)^\circ$$

as triangulated categories.

Proof. If A is an AS-regular Koszul algebra, then A^\perp is a Frobenius Koszul algebra by [21, Proposition 5.10]. It follows that $\underline{\text{grmod}} A^\perp$ itself is a triangulated category, so $\mathcal{S}(\underline{\text{grmod}} A^\perp) \cong \underline{\text{grmod}} A^\perp$ as triangulated categories. Since $*$: $\underline{\text{grmod}} A^\perp \rightarrow \underline{\text{grmod}}(A^\perp)^\circ$ is a duality as triangulated categories, the result follows. \square

The above theorem also gives another interesting example of a Cohen–Macaulay triangulated quasi-scheme.

Corollary 5.5. *Let A be a Koszul connected graded algebra over k such that both A and A^\perp are noetherian and having balanced dualizing complexes so that $Y = \mathcal{D}(\text{Proj } A^\perp) = (D^b(\text{tails } A^\perp), [1], \mathcal{A}^\perp)$ is a Cohen–Macaulay triangulated quasi-scheme over k with the canonical object ω_Y . Then $X = (\mathcal{S}(\underline{\text{grmod}} A), \Omega^{-1}, F(\omega_Y))$ is also a Cohen–Macaulay triangulated quasi-scheme with the canonical object $\omega_X = k$.*

Proof. Since $F(\mathcal{A}^\perp) = k$, this follows from Lemma 2.6 and Theorem 5.3. \square

Recall that if A is an AS-Gorenstein algebra of $\text{id}(A) = d$ such that $\omega_A \cong A(-\ell)$ in $\text{grmod } A$ for some integer ℓ , then the classical Serre duality for $X = \text{Proj } A$ can be expressed as

$$\text{Ext}_X^i(\mathcal{A}, -) \cong \text{Ext}_X^{d-1-i}(-, \mathcal{A}(-\ell))^*$$

for all i by Example 3.9. The following result can be thought of as the classical Serre duality for stable categories.

Theorem 5.6. *If A is an AS-Gorenstein Koszul algebra of $\text{id}(A) = d$ such that $\omega_A \cong A(-\ell)$ in $\text{grmod } A$ for some integer ℓ , and A^\perp is noetherian and having the balanced dualizing complex, then $X = (\mathcal{S}(\underline{\text{grmod}} A), \Omega^{-1}, k)$ is a Gorenstein triangulated quasi-scheme with the canonical object $\omega_X \cong \Omega^{1-d}k(-\ell)$. In particular,*

$$\underline{\text{Ext}}_A^i(k, -) \cong \underline{\text{Ext}}_A^{d-1-i}(-, k(-\ell))^*$$

for all i .

Proof. If A is as above, then A^\perp is also an AS-Gorenstein Koszul algebra of $\text{id}(A) = d - \ell$ such that $\omega_{A^\perp} \cong A^\perp(\ell)$ in $\text{grmod } A^\perp$ by [14, Theorem 3.7]. By Example 3.9, $Y = \mathcal{D}(\text{Proj } A^\perp)$ is a Gorenstein triangulated quasi-scheme with the canonical object $\omega_Y \cong \mathcal{A}^\perp[d - \ell - 1](\ell)$. Since $F(\mathcal{A}^\perp) \cong k$, by Lemma 2.6 and Theorem 5.3, $X = (\mathcal{S}(\underline{\text{grmod}} A), \Omega^{-1}, k)$ is also a Gorenstein triangulated quasi-scheme with the canonical object

$$\omega_X \cong F(\mathcal{A}^\perp[-d + \ell + 1](-\ell)) \cong \Omega^{1-d}F(\mathcal{A}^\perp)(-\ell) \cong \Omega^{1-d}k(-\ell).$$

By Lemma 5.2,

$$\begin{aligned}
 \mathrm{Ext}_A^i(k, -) &\cong \mathrm{Ext}_{\mathcal{S}(\underline{\mathrm{grmod}} A)}^i(k, -) \\
 &\cong \mathrm{Ext}_{\mathcal{S}(\underline{\mathrm{grmod}} A)}^{-i}(-, \Omega^{1-d}k(-\ell))^* \\
 &= \mathrm{Ext}_{\mathcal{S}(\underline{\mathrm{grmod}} A)}^{d-1-i}(-, k(-\ell))^* \\
 &\cong \underline{\mathrm{Ext}}_A^{d-1-i}(-, k(-\ell))^*
 \end{aligned}$$

for all i . \square

Let A be an AS-Gorenstein Koszul algebra such that $A^!$ is noetherian and having the balanced dualizing complex as in the above theorem. Since $\underline{\mathrm{grmod}} A$ is not a Grothendieck category, the pair $(\underline{\mathrm{grmod}} A, k)$ is not a quasi-scheme. However, the above theorem shows that Riemann–Roch Theorem and Adjunction Formula hold for the pair $(\underline{\mathrm{grmod}} A, k)$ if we use Tate–Vogel cohomologies in the definition of intersection multiplicity. If A is Frobenius, then $\mathcal{S}(\underline{\mathrm{grmod}} A) \cong \underline{\mathrm{grmod}} A$, so our theory has more concrete applications.

Example 5.7. Let A be a Frobenius connected graded algebra over k . Since A is finite dimensional over k , $X = (\underline{\mathrm{grmod}} A, \Omega^{-1}, M)$ is a Hom-finite triangulated quasi-scheme over k for any choice of $M \in \underline{\mathrm{grmod}} A$. Moreover, by [10, Proposition 8] (see also [15]), $\underline{\mathrm{grmod}} A$ has a Serre functor $\bar{\Omega} \circ \mathcal{N} : \underline{\mathrm{grmod}} A \rightarrow \underline{\mathrm{grmod}} A$ where $\mathcal{N}(-) = \bigoplus_{m=-\infty}^{\infty} \mathrm{Hom}_A(-, A(m))^* : \underline{\mathrm{grmod}} A \rightarrow \underline{\mathrm{grmod}} A$ is the graded Nakayama equivalence. Presumably, X often has finite cohomological dimension for any choice of $M \in \underline{\mathrm{grmod}} A$ so that X is a Gorenstein triangulated quasi-scheme. This is indeed the case when A is Koszul and $A^!$ is noetherian because in this case, $A^!$ is AS-regular and tails $A^!$ has finite homological dimension.

Example 5.8. On the other hand, if A is a Frobenius (ungraded) algebra over k , then $X = (\underline{\mathrm{mod}} A, \Omega^{-1}, M)$ is a Hom-finite triangulated quasi-scheme having a Serre functor $\bar{\Omega} \circ \mathcal{N} : \underline{\mathrm{mod}} A \rightarrow \underline{\mathrm{mod}} A$ where $\mathcal{N}(-) = \mathrm{Hom}_A(-, A)^* : \underline{\mathrm{mod}} A \rightarrow \underline{\mathrm{mod}} A$ is the Nakayama equivalence. However, it could be true that X always has infinite cohomological dimension for any choice of $0 \neq M \in \underline{\mathrm{mod}} A$ as explained below.

The following definition was made in [8].

Definition 5.9. Let A be a ring. For $M, N \in \underline{\mathrm{mod}} A$, we define

$$e_A(M, N) := \sup\{i \mid \mathrm{Ext}_A^i(M, N) \neq 0\}.$$

We say that A is an AB-ring if

$$\sup\{e_A(M, N) \mid M, N \in \underline{\mathrm{mod}} A \text{ such that } e_A(M, N) < \infty\} < \infty.$$

In [8], Huneke and Jorgensen showed that all noetherian commutative local complete intersection rings are AB-rings [8, Corollary 3.4], and posed a question if all

noetherian commutative local Gorenstein rings are AB-rings. Their question can be reduced to the question if all commutative local quasi-Frobenius rings are AB-rings by [8, Proposition 3.2].

Lemma 5.10. *Let A be a Frobenius algebra over k . If A is an AB-ring, then $X = (\underline{\text{mod}} A, \Omega^{-1}, M)$ has infinite cohomological dimension for any choice of $0 \neq M \in \underline{\text{mod}} A$.*

Proof. Since A is Frobenius,

$$\text{Ext}_A^i(M, -) = \underline{\text{Ext}}_A^i(M, -)$$

for all $i \geq 1$. If X has finite cohomological dimension, then, for each $N \in \text{mod } A$, $e_A(M, N) < \infty$. Since A is an AB-ring, there exists an integer m such that $e_A(M, N) \leq m$ for all $N \in \text{mod } A$. It follows that M has finite projective dimension. Since A is Frobenius, M is projective, so $M \cong 0$ in $\underline{\text{mod}} A$. \square

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